1 MIDI note numbers vs. frequency

1.1 Frequency for a given MIDI note number

Let $n \in \mathbb{R}$ be a MIDI note number, where 60.0 is taken to be C4 (middle C). Then the corresponding frequency f in Hz is given by

$$f = 440 \cdot 2^{\frac{n-69}{12}} \tag{1}$$

1.2 MIDI note number for a given frequency

Let $f \in \mathbb{R}_{>0}$ be a frequency in Hz. Then the corresponding MIDI note number n is given by

$$n = 12\left(\log_2 f - \log_2 440\right) + 69 \tag{2}$$

1.3 Equal-tempered interval between two frequencies

Let $f_1, f_2 \in \mathbb{R}_{>0}$ be frequencies in Hz. Then the 12EDO equal-tempered interval i_s (in semitones) between the two frequencies is given by

$$i_s = 12\left(\log_2 f_2 - \log_2 f_1\right) \tag{3}$$

1.4 Frequency ratio from equal-tempered interval

Let $i_s \in \mathbb{R}$ be a 12EDO equal-tempered interval in semitones, and let $n_1, n_2 \in \mathbb{R}$ be MIDI note numbers. Then the frequency ratio r between the two notes is given by

$$r = 2^{\frac{i_s}{12}} \tag{4}$$

In other words, if you multiply the first note n_1 by the ratio r, you will get the second note n_2, which is i_s semitones away from n_1.

Alternatively, if you simply wish to calculate directly from the MIDI note numbers,

$$r = 2^{\frac{n_2-n_1}{12}} \tag{5}$$

2 Equal-tempered string fingering positions

2.1 String fingering position given the number of semitones above the fundamental frequency of the string

This is the formula for the fingering position p_{string} along a string of any stringed instrument (violin, viola, etc.), given the number of 12EDO equal-tempered semitones $s \in \mathbb{R}$ above the fundamental frequency of the string.

We define the fingering position $p_{string} \in [0, 1)$ as the fraction of the string length that is not activated by the bow or plectrum. p_{string} is equal to 0 if the string is not fingered (i.e., it is an open string). p_{string} is approximately equal to $\frac{1}{4}$ if you finger the note a fourth above the open string, and is exactly equal to $\frac{1}{2}$ if you finger the note an octave above the open string.

If you know the length ℓ_{string} of the string, you can multiply it by p_{string} to get the distance in units such as inches or cm from the end of the string (where a distance of 0 means the string is vibrating for its full length, and a distance of ℓ_{string} means that no part of the string is vibrating).

$$p_{string} = 1 - \frac{1}{2^{\frac{s}{12}}} \tag{6}$$

2.2 Number of semitones above the fundamental frequency of the string, given the string fingering position

This is a formula that lets you convert harmonic nodes (or any string fingering position) into the number of 12EDO equal-tempered semitones $s \in \mathbb{R}$ above the fundamental frequency of a string $f_0 \in \mathbb{R}_{>0}$, giving you a reference point for where to finger the string to produce that harmonic.

If you know the fingering position p_{string} (such as $p_{string} = \frac{1}{4}$ or $p_{string} = \frac{3}{4}$ for the fourth harmonic at 2 octaves above the fundamental), you can plug it into this formula to get the corresponding number of equal-tempered semitones above the fundamental frequency.

If you would like to get the MIDI note number corresponding to s, you can simply add s to the MIDI note number for the string fundamental frequency.

Let $p_{string} \in [0, 1)$ be the fraction of the string length where the string is fingered. Then s is the number of equal-tempered semitones above the fundamental frequency of the string, corresponding to this fingering position:

$$s = -12 \log_2 (1 - p_{string}) \tag{7}$$

For example, say we want to know the fingering position for the fourth harmonic of the D string on a violin. In this case, we are fingering $\frac{1}{4}$ of the length of the string, so we plug $-12\log_2(1-0.25)$ into WolframAlpha or some other calculation software, and the answer is approximately 4.98044999 semitones. (This is why fingering a fourth above the open string produces the fourth harmonic, two octaves higher than f_0.)

3 Formulas for basic waveforms

3.1 Sine wave

Let $f \in \mathbb{R}_{>0}$ be the frequency in Hz. Let $\phi_0 \in [0, 2\pi)$ be the initial phase. Let A be the amplitude of the waveform. Let $t \in \mathbb{R}_{\geq 0}$ be time in seconds. Then the instantaneous amplitude of the continuous waveform at time t is given by

$$a_{sine}(t) = A\sin\left(2\pi ft + \phi_0\right) \tag{8}$$

Let $f_s \in \mathbb{R}_{>0}$ be the sampling rate in Hz, and let $n \in \mathbb{Z}_{\geq 0}$ be the sample index. Then the discrete version of the above equation is

$$a_{sine}(n) = A\sin\left(\frac{2\pi fn}{f_s} + \phi_0\right) \tag{9}$$

3.2 Sawtooth wave

Let $f \in \mathbb{R}_{>0}$ be the frequency in Hz. Let $\phi_0 \in [0, 2\pi)$ be the initial phase. Let $H \in \mathbb{N}$ be the highest harmonic index of the waveform, and let $h \in \mathbb{N}$ refer to individual harmonic indices. Let A be the amplitude of the waveform. Let $t \in \mathbb{R}_{\geq 0}$ be time in seconds. Then the instantaneous amplitude of the continuous waveform at time t is given by

$$a_{saw}(t) = A\sum_{h=1}^{H}\frac{1}{2h}\sin\left(2\pi fht + h\phi_0\right) \tag{10}$$

Let $f_s \in \mathbb{R}_{>0}$ be the sampling rate in Hz, and let $n \in \mathbb{Z}_{\geq 0}$ be the sample index. Then the discrete version of the above equation is

$$a_{saw}(n) = A\sum_{h=1}^{H}\frac{1}{2h}\sin\left(\frac{2\pi fhn}{f_s} + h\phi_0\right) \tag{11}$$

3.3 Square wave

Let $f \in \mathbb{R}_{>0}$ be the frequency in Hz. Let $\phi_0 \in [0, 2\pi)$ be the initial phase. Let $2H+1 \in \mathbb{N}$ be the highest harmonic index of the waveform, and let $2h+1 \in \mathbb{N}$ refer to individual harmonic indices. Let A be the amplitude of the waveform. Let $t \in \mathbb{R}_{\geq 0}$ be time in seconds. Then the instantaneous amplitude of the continuous waveform at time t is given by

$$a_{square}(t) = A \sum_{h=0}^{H} \frac{1}{2h+1} \sin\left((2h+1)2\pi f t + (2h+1)\phi_0\right) \tag{12}$$

Let $f_s \in \mathbb{R}_{>0}$ be the sampling rate in Hz, and let $n \in \mathbb{Z}_{\geq 0}$ be the sample index. Then the discrete version of the above equation is

$$a_{square}(n) = A \sum_{h=0}^{H} \frac{1}{2h+1} \sin\left(\frac{(2h+1)2\pi f n}{f_s} + (2h+1)\phi_0\right) \tag{13}$$

3.4 Triangle wave

Let $f \in \mathbb{R}_{>0}$ be the frequency in Hz. Let $\phi_0 \in [0, 2\pi)$ be the initial phase. Let $2H+1 \in \mathbb{N}$ be the highest harmonic index of the waveform, and let $2h+1 \in \mathbb{N}$ refer to individual harmonic indices. Let A be the amplitude of the waveform. Let $t \in \mathbb{R}_{\geq 0}$ be time in seconds. Then the instantaneous amplitude of the continuous waveform at time t is given by

$$a_{triangle}(t) = \frac{8A}{\pi^2} \sum_{h=0}^{H} \frac{(-1)^h}{(2h+1)^2} \sin\left((2h+1)2\pi f t + (2h+1)\phi_0\right) \tag{14}$$

Let $f_s \in \mathbb{R}_{>0}$ be the sampling rate in Hz, and let $n \in \mathbb{Z}_{\geq 0}$ be the sample index. Then the discrete version of the above equation is

$$a_{triangle}(n) = \frac{8A}{\pi^2} \sum_{h=0}^{H} \frac{(-1)^h}{(2h+1)^2} \sin\left(\frac{(2h+1)2\pi f n}{f_s} + (2h+1)\phi_0\right) \tag{15}$$

4 Formulas for envelopes

4.1 Linear envelopes

A linear envelope takes the form

$$f(x) = mx + b \tag{16}$$

where $m \in \mathbb{R}$ is the slope of the envelope and $b \in \mathbb{R}$ is the y-intercept. To determine a suitable m, given points $(x_1, y_1), (x_2, y_2)$, the following formula can be used:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{17}$$

4.2 Generalized n-th power envelopes

A curved n-th power envelope takes the form

$$f(x) = m(x - a)^n + b \tag{18}$$

where $m \in \mathbb{R}$ is a gain coefficient of the envelope, $n \in \mathbb{R}_{\geq 1}$ is the curve exponent, a is the x-offset, and $b \in \mathbb{R}$ is the y-intercept. Linear envelopes are a form of the generalized n-th power envelope where $n = 1$. If $m = 0$, the envelope is a straight horizontal line of the formula $y = b$.

For $m \neq 0$ and $n > 1$, the curve can take four different shapes:

1. Convex, rising from $x = 0$ $(m > 0)$
2. Convex, falling from $x = 0$ $(m > 0)$
3. Concave, rising from $x = 0$ $(m < 0)$
4. Concave, falling from $x = 0$ $(m < 0)$

The formula for determining a suitable m is similar, but not quite the same, for the four different formulas above. In both of the following formulas, b is the y-intercept, and n is the curve exponent. We assume that we know the values corresponding to $x = 0$ and $x = x_1 > 0$.

1. Convex, rising from $x = 0$, **or** concave, falling from $x = 0$: Let $b \in \mathbb{R}$ be the desired y-value at input $x = 0$, and let $c \in \mathbb{R}$ be the desired y-value at input $x = x_1 > 0$. Then the formula for the gain coefficient m is given by

   $$m = \frac{c - b}{x_1^n} \tag{19}$$

   and $a = 0$.

2. Convex, falling from $x = 0$, **or** concave, rising from $x = 0$: Let $c \in \mathbb{R}$ be the desired y-value at input $x = 0$, and let $b \in \mathbb{R}$ be the desired y-value at input $x = x_1 > 0$. Then the formula for the gain coefficient m is given by

   $$m = \frac{c - b}{(-x_1)^n} \tag{20}$$

   and $a = x_1$